Global Structure of Certain Static Spacetimes (I)

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Abstract

In this paper, static spacetimes with a topological structure of $\mathbb{R}^2 \times \mathcal{N}$ is studied, where $\mathcal{N}$ is an arbitrary manifold. Well known Schwarzschild spacetime and Reissner-Nordström spacetime are special cases. It is shown that the existence of a constant and positive surface gravity κ ensures the existence of the Killing horizon, with the cross section homeomorphic to $\mathcal{N}$.

1 Introduction

In general relativity, specific spacetimes often tell us much information that is related to the properties of more generic spacetimes. The rather simple spacetimes, if the trivial Minkowski spacetime is excluded, should be the spherically symmetric static spacetimes: the sourceless Schwarzschild spacetime, the Reissner-Nordström spacetime with electromagnetism, and so on. Simple as they are, they have already possessed variety of features which remain to be true for generic static, or, even stationary, spacetimes. For example, if the stationary spacetime possesses more structures such as additional symmetries or else, there is always the Regge-Wheeler tortoise function encountered in these spacetimes and we find it useful in various discussions: For the Kruskal extensions[1, 2, 3, 4], it is the start point where the ingoing and outgoing Eddington-Finkelstein coordinates and, thereafter, the Kruskal-Szekeres coordinates, are introduced; In probing the thermodynamical properties of black holes, the tortoise function is also widely used (see, for example, [5, 6]). In string theories, situations are similar.

Detailed discussion of specific spacetimes benefits us a lot. But restricting our attention to particular spacetimes often sinks us into trouble: Sometimes a conclusion is so directive to be derived that deep sense of the problem will be ignored. It just like that you pave your way directly to your destination, while wonderful scenery around have been missed because you are so quick to reach where you want to go. The worst thing is that, since specific spacetimes often possess additional structures such as the spherical symmetry (namely, the $SO(3)$ symmetry)[7, 8], compactness of the cross section of the Killing horizon and so on, we don't know whether certain conclusion is closely related to these properties. Example of this situation may be the first law of black hole mechanics[3, 9, 10]. In the ordinary cases, the entropy S of the black hole is believed to be proportional to the area of the cross section of the horizon. But, what if the cross section of the Killing horizon is not compact, with the area being infinite? The proof of the constancy of the

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surface gravity is another example of depending on the additional symmetries of the spacetime. To prove it, one has to prove that the surface gravity on the bifurcate sphere is constant. For a sphere, this is certainly true. But, if the cross section of the Killing horizon is not a sphere, or, if the Killing horizon has no spherical symmetry, does the total proof remain valid?

In this paper we do not want to aim at any particular object. We just consider static spacetimes of arbitrary dimension $d \geq 3$ which, topologically, is the Cartesian product of $\mathbb{R}^2$ and an arbitrary manifold $\mathcal{N}$ endowed with a Riemannian metric $\mathbf{h}_{\mathcal{N}}$. We don't assume any symmetry about the Riemannian manifold $(\mathcal{N}, \mathbf{h}_{\mathcal{N}})$. This, of course, has included Schwarzschild spacetime and Reissner-Nordström spacetime as special examples. We try to establish a somehow generic formulism for the extendibility of such spacetimes and outline the extendibility of the spacetime as in the following:

The static spacetime admits the Killing vector field $\mathbf{K}$ as a globally defined smooth vector field, which defines a globally smooth function $g_{tt} = \mathbf{g}(\mathbf{K}, \mathbf{K})$. The event horizon, if there is, consists of the zero points of this function, implying that $\mathbf{K}$ is lightlike on it. The function g_{tt} is intrinsic, independent of the choice of any coordinates. There is also the intrinsic smooth function, the Regge-Wheeler tortoise function r_*. But it is, generally speaking, not globally defined. The maximal domain of this smooth function is called a region of the spacetime. If the closure of its maximal domain can't cover the spacetime, namely, not the whole spacetime itself, there must be another Regge-Wheeler tortoise function whose maximal domain has no common subset with the domain of r_*. If the closures of these domains still can not cover the whole spacetime, there must be a third tortoise function, and so on. Then finally we have a set of tortoise functions as well as their corresponding maximal domains (regions, as we call). Any pair of these regions have no common subsets except for the empty set. Between a pair of adjacent regions there is an event horizon. In this situation, each of such a region is called extendible. There is the possibility that the whole spacetime is extendible. If the surface gravity is a nonzero constant on a horizon, and if the tortoise function tends to infinite as it approaches the horizon, we can give the formulism showing how it can be extended, which is quite a similar version of the Kruskal extension.

The paper is organized as the following. In §2, we state the structure of the static spacetime whose extendibility would be considered. In §3 we give the coordinate description to the region in which the Killing vector field $\mathbf{K}$ is either timelike or spacelike. Two special lightlike vector fields are given for future use. In §4 we investigate the completeness of the integral curves of the above two lightlike vector fields so that we can trace the behavior of the tortoise function as it approaches the zero points of g_{tt}. In §5 we derive the formulism as how the region can be extend out of the horizon. And §6 is the closure part of this paper, the discussion and conclusions.

2 The Structure of Spacetimes Considered Here

In this paper, the spacetime we considered is a connected static spacetime $(\mathcal{M}, \mathbf{g})$ of $d \geq 3$ dimensional, where $\mathbf{g}$ is the metric tensor of the spacetime with the signature of $\mathrm{diag}(-1, 1, \ldots, 1)$. The Killing vector field is denoted by $\mathbf{K}$. Hence $\mathbf{K}$ satisfies two equations, the Killing equation

$$\mathcal{L}_{\mathbf{K}}\, \mathbf{g} = 0 \tag{1}$$

and the Fröbenius condition

$$\tilde{\mathbf{K}} \wedge \mathrm{d}\tilde{\mathbf{K}} = 0. \tag{2}$$

Note that, in the above equation, $\tilde{\mathbf{K}}$ is the 1-form resulted by raising the index of $\mathbf{K}$ with $\mathbf{g}$: In the language of abstract indices[3, 11], $\mathbf{K}$ is denoted by K^a, and $\tilde{\mathbf{K}}$ is denoted by $K_a = g_{ab}K^b$.

To simplify our problem, we shall assume further that the spacetime $\mathcal{M}$ is a Cartesian product of $\mathbb{R}^2$ and $\mathcal{N}$, where $\mathcal{N}$ is an arbitrary $d-2$ dimensional manifold endowed with a Riemannian

metric $\mathbf{h}_{\mathcal{N}}$. We assume that the Killing vector field $\mathbf{K}$ is always "tangent" to the directions of $\mathbb{R}^2$. That is, let $\pi : \mathcal{M} \longrightarrow \mathcal{N}$ be the projection from $\mathcal{M} = \mathbb{R}^2 \times \mathcal{N}$ to $\mathcal{N}$, then $\pi_*(\mathbf{K}|_p) = 0$ at any point p in $\mathcal{M}$. If we define the symmetric tensor field $\mathbf{h}$ on $\mathcal{M}$ to be

$$\mathbf{h} = \pi^* \mathbf{h}_{\mathcal{N}}, \tag{3}$$

the Killing vector field $\mathbf{K}$ will be orthogonal to the symmetric tensor field $\mathbf{h}$ on $\mathcal{M}$. That is, if the 1-forms $h_{ab}K^b$ and $K^b H_{ba}$ are denoted by $\mathbf{h} \cdot \mathbf{K}$ and $\mathbf{K} \cdot \mathbf{h}$, respectively, we could claim that

$$\mathbf{h} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{h} = 0. \tag{4}$$

In order to make the calculations easier, we may introduce local coordinates (t, r, x^i) ($i = 1, \ldots, d-2$) into certain region of $\mathcal{M}$, with t and r along the two directions of $\mathbb{R}^2$ and $x^i = \pi^* x^i_{\mathcal{N}}$ being the pull back of local coordinate $x^i_{\mathcal{N}}$ on $\mathcal{N}$. Now we can write $\mathbf{K}$ and $\mathbf{g}$ as

$$\mathbf{K} = K_t \frac{\partial}{\partial t} + K_r \frac{\partial}{\partial r}$$

and

$$\mathbf{g} = g_{tt}\, \mathrm{d}t \otimes \mathrm{d}t + g_{tr}\, (\mathrm{d}t \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}t) + g_{rr}\, \mathrm{d}r \otimes \mathrm{d}r + \beta \otimes \mathrm{d}r + \mathrm{d}r \otimes \beta + C\, \mathbf{h}, \tag{5}$$

where $\beta = B_i\, \mathrm{d}x^i$ is a differential 1-form on $\mathcal{M}$. According to Eqs.(3) and (4), $\mathbf{h}$ can be expressed as

$$\mathbf{h} = h_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

where the functions h_{ij}'s are independent of the coordinates t and r. Obviously, the Lie derivative of $\mathbf{h}$ with respect to $\mathbf{K}$ is zero.

In the neighborhood of a point at which $\mathbf{K}$ is nonzero, we can always choose its neighborhood properly so that $\mathbf{K}$ is nonzero everywhere in it. When the coordinate system (t, r, x^i) is defined in this neighborhood, we can set t and r carefully chosen in order that $K_t = 1$, $K_r = 0$ in the above equation: $\mathbf{K} = \frac{\partial}{\partial t}$. Since the spacetime is static, coordinates can be chosen such that either g_{tt} or g_{tr} is zero. In this paper, our topic is mainly focused on regions where $\mathbf{K}$ is nonzero everywhere.

3 In the Region Where K Is Timelike or Spacelike

First let us consider the coordinate neighborhood in which $\mathbf{K}$ is either timelike or spacelike. In such a neighborhood, the coordinates t and r can be organized so that $g_{tr} = g_{rt} = 0$. Then Eq.(1) implies that the functions g_{tt}, g_{rr}, $g_{tr} = g_{rt}$, B_i and C are all independent of t.

In this case, a coordinate transformation $(t, r, x^i) \longrightarrow (t, r_*, x^i)$ can be used for simplifying the discussion. Here

$$r_* = r_*(r, x) \tag{6}$$

is a function that will turn the metric $\mathbf{g}$ into the form of

$$\mathbf{g} = g_{tt}\, (\mathrm{d}t \otimes \mathrm{d}t - \mathrm{d}r_* \otimes \mathrm{d}r_*) + \beta_* \otimes \mathrm{d}r_* + \mathrm{d}r_* \otimes \beta_* + \mathbf{h}_*. \tag{7}$$

A bit of calculation reveals that

$$\frac{\partial r_*}{\partial r} = \sqrt{-\frac{g_{rr}}{g_{tt}}}, \tag{8}$$

$$\beta_* = \sqrt{-\frac{g_{tt}}{g_{rr}}}\beta + g_{tt}\frac{\partial r_*}{\partial x^i}\mathrm{d}x^i, \tag{9}$$

$$\mathbf{h}_* = C\, \mathbf{h} - g_{tt}\frac{\partial r_*}{\partial x^i}\frac{\partial r_*}{\partial x^j}\mathrm{d}x^i \otimes \mathrm{d}x^j - \frac{\partial r_*}{\partial x^i}\sqrt{-\frac{g_{tt}}{g_{rr}}}\, (\beta \otimes \mathrm{d}x^i + \mathrm{d}x^i \otimes \beta). \tag{10}$$

It is easy to see that $\mathbf{h}_*$ is the induced metric of $\mathbf{g}$ on the surface determined by t = constant and r_* = constant. When we turn back to the well known cases such as the Reissner-Nordström spacetime, the function r_* is just the Regge-Wheeler tortoise coordinate function. Since the Lie derivatives of β and $\mathbf{h}$ with respect to $\mathbf{K}$ are zero, we can see that

$$\mathcal{L}_{\mathbf{K}}\,\beta_* = 0, \qquad \mathcal{L}_{\mathbf{K}}\,\mathbf{h}_* = 0 \tag{11}$$

with a peer and, still, $\beta_*(\mathbf{X}) = 0$, $\mathbf{h}_*(\mathbf{X}, \mathbf{Y}) = 0$ whenever $\mathbf{X}$ satisfies $\pi_*\mathbf{X} = 0$.

Now we want to find out that whether there are vector fields $\mathbf{X}_+$ and $\mathbf{X}_-$ on $\mathcal{M}$ satisfying:

(1) They lie in the $\mathbb{R}^2$ plains, namely, $\pi_*\mathbf{X}_\pm = 0$;

(2) They are lightlike vector fields, namely, $\mathbf{g}(\mathbf{X}_+, \mathbf{X}_+) = \mathbf{g}(\mathbf{X}_-, \mathbf{X}_-) = 0$;

(3) They are geodesic vector fields, namely, $\nabla_{\mathbf{X}_+}\mathbf{X}_+ = 0$ and $\nabla_{\mathbf{X}_-}\mathbf{X}_- = 0$.

Using the coordinate (t, r_*, x^i), we can determine these vector fields, if there exists, to be

$$\mathbf{X}_\pm = X\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial r_*}\right),$$

according to the first two conditions. It is not necessary to calculate the Christoffel symbols for $\nabla_{\mathbf{X}_\pm}\mathbf{X}_\pm$. In fact, there is the formula[1] $\mathbf{X}_+ \cdot \mathcal{L}_{\mathbf{X}_+}\mathbf{g} = \nabla_{\mathbf{X}_+}\tilde{\mathbf{X}}_+ + (\nabla\tilde{\mathbf{X}}_+)\cdot\mathbf{X}_+ = \nabla_{X_+}\tilde{\mathbf{X}}_+$, and we know that the Lie derivatives are more easier to calculate. It is easy to verify that

$$\nabla_{\mathbf{X}_\pm}\tilde{\mathbf{X}}_\pm = X\left[\frac{\partial}{\partial t}(g_{tt}X) \pm \frac{\partial}{\partial r_*}(g_{tt}X)\right](\mathrm{d}t \mp \mathrm{d}r_*) \pm \mathcal{L}_{\mathbf{X}_\pm}(X\beta_*)$$

We can set $X = \frac{1}{|g_{tt}|}$, yielding

$$\mathbf{X}_\pm = \frac{1}{|g_{tt}|}\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial r_*}\right), \tag{12}$$

$$\nabla_{\mathbf{X}_\pm}\tilde{\mathbf{X}}_\pm = \pm\mathcal{L}_{\mathbf{X}_\pm}\left(\frac{1}{|g_{tt}|}\beta_*\right) = \pm i_{\mathbf{X}_\pm}\mathrm{d}\left(\frac{1}{|g_{tt}|}\beta_*\right). \tag{13}$$

Notice that the function r_* is not uniquely determined by Eq.(8). For a known function $r_* = r_*(r, x)$ satisfying this equation, one can always give another function $r'_* = r_* - \psi(x)$ where $\psi(x)$ is an arbitrary function of x^i's only. If we define β'_* for r'_* as that in Eq.(9) for r_*, then

$$\mathrm{d}\psi = \frac{1}{|g_{tt}|}(\beta_* - \beta'_*) \tag{14}$$

is a locally exact 1-form. If $\frac{1}{|g_{tt}|}\beta_*$ itself is exact, we can choose $\psi(x)$ properly so that $\beta'_* = 0$. This makes the 1-form $\frac{1}{|g_{tt}|}\beta_*$ significant. In fact, it needs not necessarily to be exact to let the vectors $\mathbf{X}_\pm$ be geodesic, for example, $\frac{1}{|g_{tt}|}\beta_*$ can be just closed, or else, $\frac{1}{|g_{tt}|}\beta_* = B_{*i}\,\mathrm{d}x^i$ with the coefficients B_{*i} independent of t and r_*.

In the following, we will assume that $\frac{1}{|g_{tt}|}\beta_*$ is an exact 1-form. As discussed in the above paragraph, it is equivalent to assume that $\beta_* = 0$. This has covered the well known Schwarzschild spacetime and Reissner-Nordström spacetime as special cases: In the case of Reissner-Nordström spacetimes, the Riemannian manifold $\mathcal{N}$ is a 2-dimensional sphere. The metric components g_{tt} and g_{rr} are independent of $(x^1, x^2) = (\theta, \varphi)$ with the 1-form β being zero, making the Regge-Wheeler tortoise coordinate function r_* to depend only on r and $\beta_* = 0$. In the following discussion, we

[1] Due to the well known formula, in the language of abstract indices, $\mathcal{L}_{\mathbf{X}}\,g_{ab} = \nabla_a X_b + \nabla_b X_a$.

do not need g_{tt} and g_{rr} to be independent of x^i, and we do not need β to be zero, either. What we need is just that a function $r_* = r_*(r,x)$ can be picked making the 1-form β_* to be zero.

Immediately, the 1-forms

$$\tilde{\mathbf{X}}_+ = \eta_{tt}\,\mathrm{d}u, \qquad \tilde{\mathbf{X}}_- = \eta_{tt}\,\mathrm{d}v \tag{15}$$

are (at least locally in the neighborhood where the coordinates t, r and x^i are defined) exact, where η_{tt} is the sign of g_{tt}, and $u = t - r_*$ and $v = t + r_*$ are the outgoing and ingoing lightlike coordinate functions, respectively. In the coordinate neighborhood, the component form of the metric $\mathbf{g}$ in Eq.(5) is well defined. The question is, what would happen when it tends to the boundary of the coordinate neighborhood.

If the symmetric tensor field $\mathbf{h}_*$ tends to be degenerate or undefinable at the boundary of the coordinate neighborhood, we can believe that the spacetime could not extend out of the boundary any more. If, however, only the function g_{tt} tends to be zero or infinity at the boundary, the conclusion may be quite different. What it will be depends on the coordinate free expressions. This is why we want to introduce the two lightlike geodesic vector fields $\mathbf{X}_+$ and $\mathbf{X}_-$.

4 The Geodesic Curves of $\mathbf{X}_\pm$

As we have seen, there are the relations

$$\mathbf{X}_+ u = \mathrm{d}u\,(\mathbf{X}_+) = \eta_{tt}\,\tilde{\mathbf{X}}_+(\mathbf{X}_+) = \eta_{tt}\,\mathbf{g}(\mathbf{X}_+,\mathbf{X}_+) = 0, \qquad \mathbf{X}_- v = \eta_{tt}\,\tilde{\mathbf{X}}_-(\mathbf{X}_-) = 0 \tag{16}$$

and

$$\mathbf{X}_+ v = \mathrm{d}v(\mathbf{X}_+) = \eta_{tt}\,\tilde{\mathbf{X}}_-(\mathbf{X}_+) = \frac{2}{|g_{tt}|}, \qquad \mathbf{X}_- u = \frac{2}{|g_{tt}|}. \tag{17}$$

The meaning of these relations can be interpreted as in the following. Suppose $\gamma_+ : I \longrightarrow \mathcal{M}$, $\lambda \longmapsto \gamma_+(\lambda)$ is an integral curve of the lightlike vector $\mathbf{X}_+$, where I is some interval of $\mathbb{R}$ containing 0 and $\gamma_+(0) = p$ lies in the coordinate neighborhood. Obviously γ_+ is a geodesic lightlike curve which can be described, in the coordinate language, as

$$u(\lambda) = u = \text{constant},\ \ v(\lambda) = v(\gamma_+(\lambda)),\ \ x^i(\lambda) = x^i = \text{constant}.$$

On the one hand, the tangent vector $\dot{\gamma}_+(\lambda)$ can be written in the coordinate system (u, v, x^i) as

$$\dot{\gamma}_+(\lambda) = \frac{\mathrm{d}v}{\mathrm{d}\lambda}\frac{\partial}{\partial v}\bigg|_{\gamma_+(\lambda)} = \frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}\lambda}\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}\Big) = \frac{|g_{tt}|}{2}\frac{\mathrm{d}v}{\mathrm{d}\lambda}\,\mathbf{X}_+\bigg|_{\gamma_+(\lambda)}.$$

On the other hand, $\dot{\gamma}_+(\lambda)$ is nothing else but $\mathbf{X}_+\Big|_{\gamma_+(\lambda)}$. Hence we obtain

$$\frac{\mathrm{d}v}{\mathrm{d}\lambda} = \frac{2}{|g_{tt}(\frac{v(\lambda)-u}{2}, x)|}. \tag{18}$$

Combined with $\frac{\mathrm{d}u}{\mathrm{d}\lambda} = 0$, it reveals that

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\mathrm{d}r_*}{\mathrm{d}\lambda} = \frac{1}{|g_{tt}(\frac{v(\lambda)-u}{2}, x)|}. \tag{19}$$

For a pair of affine parameters λ_0 and λ_1, let $v_0 = v(\lambda_0)$ and $v_1 = v(\lambda_1)$, then

$$v_1 - v_0 = \int_{\lambda_0}^{\lambda_1} \frac{2}{|g_{tt}(r_*(\lambda), x)|}\,\mathrm{d}\lambda, \tag{20}$$

$$\lambda_1 - \lambda_0 = \frac{1}{2}\int_{v_0}^{v_1} |g_{tt}(\frac{v-u}{2}, x)|\,\mathrm{d}v = \int_{\frac{v_0-u}{2}}^{\frac{v_1-u}{2}} |g_{tt}(r_*, x)|\,\mathrm{d}r_*. \tag{21}$$

Similarly, for the integral curve $\gamma_- : I \longrightarrow \mathcal{M}, \lambda \longrightarrow \gamma_-(\lambda)$ which is described by

$$u(\lambda) = u(\gamma_-(\lambda)),\ v(\lambda) = v = \text{constant},\ x^i(\lambda) = x^i = \text{constant},$$

we can obtain

$$\frac{\mathrm{d}u}{\mathrm{d}\lambda} = \frac{2}{|g_{tt}(\frac{v-u(\lambda)}{2}, x)|} \tag{22}$$

and, for a pair of affine parameters λ_0 and λ_1 with $u_0 = u(\lambda_0)$ and $u_1 = u(\lambda_1)$,

$$u_1 - u_0 = \int_{\lambda_0}^{\lambda_1} \frac{2}{|g_{tt}(\frac{v-u(\lambda)}{2}, x)|}\,\mathrm{d}\lambda, \tag{23}$$

$$\lambda_1 - \lambda_0 = \frac{1}{2}\int_{u_0}^{u_1} |g_{tt}(\frac{v-u}{2}, c)|\,\mathrm{d}u = \int_{\frac{v-u_1}{2}}^{\frac{v-u_0}{2}} |g_{tt}(r_*, x)|\,\mathrm{d}r_*. \tag{24}$$

These two sets of curves are closely related to each other. Let $v = S(\lambda, u, x)$ be a solution of the differential equation (18), then it is very easy to verify that

$$u(\lambda) = 2v - S(-\lambda, v, x) \tag{25}$$

is a solution to the equation (22). Both $v(\lambda)$ and $u(\lambda)$ are increasing functions of λ.

If the function g_{tt} tends to zero along one of the curves when the affine parameter λ increases (or decreases), we may ask whether the parameter λ is finite or not before it reaches the zero point of g_{tt}. If it is, we call that the geodesic is incomplete in that direction. Otherwise we say it is complete in that direction. Take the direction in which λ increases, for example. If the geodesic is complete in that direction, then λ tends to $+\infty$ as it approaches the "zero point" of g_{tt}. Then Eq.(19) indicates that this "point", if it exists in the spacetime, corresponds to $r_* = +\infty$. We just call naïvely that the "zero point" of g_{tt} "is located at" where $r_* = +\infty$.

As for incomplete geodesics, cases will be complicated. Suppose that g_{tt} tends to zero along the curve of $\mathbf{X}_+$ as λ tends to a finite value λ_m. If the velocity at which g_{tt} tends to zero is not slower than that of $\frac{1}{\lambda-\lambda_\mathrm{m}}$, the "zero point" of g_{tt} is still located at where $r_* = +\infty$, or accurately speaking, at where $v = +\infty$ (cf Eqs.(19) and (20)). And, because the relationship (25) of the integral curves of $\mathbf{X}_+$ and $\mathbf{X}_-$, we can safely assert that one the geodesic of $\mathbf{X}_-$ will be incomplete as the parameter increases, and that the "zero point" of g_{tt} is located at where $u = -\infty$, and *vice versa*. As we have known, "locations" where $u = -\infty$ and where $v = +\infty$ are not the same. Both of these locations give $r_* = +\infty$. The most miserable thing is that the geodesics are incomplete and leading g_{tt} tends to zero slower than or as quickly as $\frac{1}{\lambda-\lambda_\mathrm{m}}$, which locates the "zero points" of g_{tt} at where $r_* = \rho(x)$ for some function $\rho(x)$. Topic like this will be left to proceeding paper. In this paper, we discuss cases in which the "zero points" of g_{tt} are located at where $r_* = \infty$ only.

The Schwarzschild spacetime and the Reissner-Nordström spacetime match the above discuss. The latter has regions in which every geodesic is incomplete in both directions, giving the location of $r_* = +\infty$ as well as $r_* = -\infty$. As in these examples, generally speaking, the spacetime can be extended far away out of the coordinate neighborhood. In the following, we shall concentrate our attention to discuss it.

But before we go ahead, we had better give a summary to the above discussion.

Notice that the metric tensor $\mathbf{g} = g_{tt}\,(\mathrm{d}t \otimes \mathrm{d}t - \mathrm{d}r_* \otimes \mathrm{d}r_*) + \mathbf{h}_*$ is rather simple, where we have assumed that $\mathbf{h}_*$ behaves very well, no matter in or out of the coordinate neighborhood. And g_{tt} may encounter some zero points on the boundary of this neighborhood. Experience in studying Schwarzschild spacetime or Reissner-Nordtröm spacetime implies that these zero points are often not the singularities of the whole spacetime. They are often believed to be due to the choice of the coordinates. It is somewhat right but not the case, in fact. Why? Because the function g_{tt}

has its own significance as the square norm of the Killing vector field $\mathbf{K}$, $\mathbf{g}(\mathbf{K},\mathbf{K})$. In this sense it stands there independent of how the coordinates have been chosen. Thus, to treat its zero points as the illness of the coordinates, as that is misunderstood by quite a large amount of people, is unfair for the coordinates. Zero points of g_{tt}, were they not the singularities, are really special points of the spacetime that must be considered especially.

5 The Kruskal Extension

For the metric tensor $\mathbf{g}$ with the "zero points" of g_{tt} located at where $r_* = -\infty$, or equivalently, $u = +\infty$ and $v = -\infty$, we imitate what has been done in the Kruskal spacetime[12], introducing two functions

$$U = -e^{-\kappa u}, \qquad V = e^{\kappa v}, \tag{26}$$

where κ is a positive constant to be determined. They are increasing functions of u and v, respectively. And the "surfaces" $u = +\infty$ and $v = -\infty$ are described as $U = 0$ and $V = 0$, respectively. Therefore the coordinate neighborhood of system (t, r, x^i) is contained in the region $U < 0$ and $V > 0$. Using these two variables, the metric $\mathbf{g}$ looks like

$$\mathbf{g} = G(U, V, x)\,(\mathrm{d}U \otimes \mathrm{d}V + \mathrm{d}V \otimes \mathrm{d}U) + \mathbf{h}_*, \tag{27}$$

where

$$G(U, V, x) = \frac{1}{2\kappa^2}\, g_{tt}\, e^{-2\kappa r_*} \tag{28}$$

is a function of U, V and x^i. It is well known that the function r_* can be treated as a function of U and V, defined by $r_* = \frac{1}{2\kappa}\ln(-UV)$.

If there is a positive constant κ such that

$$0 < G_0 = \lim_{r_* \to -\infty} |g_{tt}(r_*, x)|\, e^{-2\kappa r_*} < +\infty, \tag{29}$$

then $\displaystyle\lim_{u\to+\infty} G(u, v, x) = \lim_{v\to-\infty} G(u, v, x) = \frac{\eta_{tt}}{2\kappa^2} G_0$ is a nonzero function of x^i's, and we are able to embed isotropically the coordinate neighborhood of system (t, r, x^i) into a region that contains points where $U = 0$ or $V = 0$. That is, the (or some of) zero points of g_{tt} are not singularities. Instead, they are points of the spacetime, or else, the spacetime can be extended to include them if they were not parts of the spacetime originally. The constant κ can be calculated, if it exists. In fact, L'Hospital's rule can be applied to Eq.(29) since g_{tt} tends to zero as we have assumed. Thus $\displaystyle G_0 = \lim_{r_*\to-\infty} \frac{\eta_{tt}}{2\kappa\, e^{2\kappa r_*}} \frac{\partial g_{tt}}{\partial r_*} = G_0 \lim_{r_*\to-\infty} \frac{1}{2\kappa\, g_{tt}} \frac{\partial g_{tt}}{\partial r_*} = \frac{G_0}{2\kappa} \lim_{r_*\to-\infty} \frac{\partial}{\partial r_*} \ln g_{tt} \neq 0$, from which we obtain the positive constant

$$\kappa = \frac{1}{2} \lim_{r_*\to-\infty} \frac{\partial}{\partial r_*} \ln|g_{tt}| = \lim_{r\to r_\mathrm{m}} \frac{1}{\sqrt{|g_{rr}|}} \frac{\partial\sqrt{|g_{tt}|}}{\partial r}, \tag{30}$$

where $r \to r_\mathrm{m}$ corresponds to $r_* \to -\infty$. The above formulas have been derived and widely used in references [6]. One can verify that, when applied to the Schwarzschild spacetime or the Reissner-Nordström spacetime, they give the resulted κ as the surface gravity. Intuitively, they reveal the demand that g_{tt} must behave much like $e^{\kappa(v-u)} = e^{2\kappa r_*}$ nearby the zero points of g_{tt}. Notice that g_{tt} and g_{rr} often depend on the coordinates x^i. But, the limit value κ in the above equation must be independent of any coordinates. This has imposed strong limitation upon the metric $\mathbf{g}$.

In the regular region, namely, where $U \neq 0$ and $V \neq 0$, the partial derivatives of G with respect to U and V are smooth. For example,

$$\frac{\partial G}{\partial U} = \frac{1}{4\kappa^2 U}\frac{\partial g_{tt}}{\partial r_*}e^{-2\kappa r_*} - \frac{1}{2\kappa^2 U} g_{tt} e^{-2\kappa r_*}. \tag{31}$$

As the variable U tends to 0^-, the limit of the above is of type $\frac{0}{0}$ for a nonzero V. Since the partial derivative of G with respect to U at $U = 0$ is

$$\left.\frac{\partial G}{\partial U}\right|_{U=0^-} = \lim_{U\to 0^-} \frac{G(U,V,x) - G(0,V,x)}{U} = \lim_{U\to 0^-}\frac{\partial G}{\partial U}$$

if the last limit exists, and hence it will be continuous at $U = 0$. If we write down the partial derivative of G with respect to V, the expression indicates that it is also continuous at $V = 0$, implying that both the partial derivatives are continuous on the whole region of $U \leq 0$ and $V \geq 0$. If all the limits of various partial derivatives of G exist as $U \to 0^-$ and $V \to 0^+$, the metric can be smoothly extended to the region which includes the zero points of g_{tt}, as what we have known to the Kruskal spacetimes.

Suppose that there is another spacetime (or region, we may call it the region II and call the spacetime region discussed in the above as the region I for convenience) that is similar to the above discussed: the region II is topologically also a Cartesian product of $\mathbb{R}^2$ and $\mathcal{N}$, endowed with a metric $\mathbf{g}'$ admitting a Killing vector field $\mathbf{K}'$ with similar properties. Suppose that the zero points of g'_{tt} are located at where $r'_* = +\infty$ this time, and the positive constant κ is the same as the above, enabling

$$U' = e^{\kappa u'}, \qquad V' = -e^{-\kappa v'} \tag{32}$$

such that

$$\mathbf{g}' = G'(U',V')(\mathrm{d}U' \otimes \mathrm{d}V' + \mathrm{d}V' \otimes \mathrm{d}U') + \mathbf{h}'_* \tag{33}$$

can be smoothly extended to include the points where $U' = 0$ as well as $V' = 0$. If, at $U' = 0$ and $V' = 0$, respectively, all the partial derivatives of G' of various order with respect to U' and V' equal to the corresponding ones of G at $U = 0$ and $V = 0$, respectively, the region I and the region II can be glued together by identifying $(U = 0, V, x)$ with $(U' = 0, V', x)$ provided that the remainder part $\mathbf{h}_*$ and $\mathbf{h}'_*$ can be fixed together smoothly. And, we know that we can copy the region I to be a new region I$'$ with the variables U and V of opposite values of those in the region I, and the region II can also be copied to be a new region II$'$ in the same way. Thus region I and region II$'$ can be glued together by identifying $(U, V = 0, x)$ with $(U', V' = 0, x)$. In the same way, regions I$'$ and II$'$, regions I$'$ and II can also be glued together, respectively. In this way, we have recovered a Kruskal spacetime, just as that has been done for the well known examples.

The expression of metric $\mathbf{g}$ in Eq.(27) indicates that hypersurfaces U = constant or V = constant are lighlike. The lightlike hypersurface $U = 0$ with $V \neq 0$ or $V = 0$ with $U \neq 0$, the event horizons, consist of zero points of $g_{tt} = \mathbf{g}(\mathbf{K},\mathbf{K})$. That is, the Killing vector field $\mathbf{K}$ becomes lighlike on these hypersurfaces while it is either timelike or spacelike off them. Using the coordinates U, V and x^i, the Killing vector field reads

$$\mathbf{K} = \kappa V \frac{\partial}{\partial V} - \kappa U \frac{\partial}{\partial U}. \tag{34}$$

Obviously, on the horizon $U = 0$, $\mathbf{K} = \kappa V \frac{\partial}{\partial V}$ is tangent to it. And it is for the same reason that $\mathbf{K} = -\kappa U \frac{\partial}{\partial U}$ is also tangent to the horizon $V = 0$. Hence the event horizons are Killing horizons, for, on these horizons, the 1-form $\tilde{\mathbf{K}}$ is still well defined. To see it, we begin from the off horizon expression

$$\tilde{\mathbf{K}} = g_{tt}\,\mathrm{d}t = \frac{g_{tt}}{2\kappa}\Big(\frac{1}{V}\mathrm{d}V - \frac{1}{U}\mathrm{d}U\Big).$$

Using the function G, which is well defined in the neighborhoods of the horizons, the above expression becomes

$$\tilde{\mathbf{K}} = \kappa\, G\,(\,V\mathrm{d}U - U\mathrm{d}V), \tag{35}$$

coinciding with Eq.(34). In order to calculate $\nabla_{\mathbf{K}}\mathbf{K}$ on the horizons, we first extend $\mathbf{K}|_{U=0}$ or $\mathbf{K}|_{V=0}$ to be a vector field defined in the neighborhood of the horizon. The extension varies arbitrarily. We simply choose $\mathbf{K}$ as that. Eq.(1), or equivalently, $\nabla_a K_b + \nabla_b K_a = 0$ in the language of abstract indices, gives the formula $\mathbf{K}\cdot\mathcal{L}_{\mathbf{K}}\,\mathbf{g} = 0 = \nabla_{\mathbf{K}}\tilde{\mathbf{K}} + \frac{1}{2}\mathrm{d}g_{tt}$. Hence

$$\nabla_{\mathbf{K}}\tilde{\mathbf{K}} = -\frac{1}{2}\mathrm{d}g_{tt} = \frac{\kappa\, G}{2g_{tt}}\frac{\partial g_{tt}}{\partial r_*}(\,U\mathrm{d}V + V\mathrm{d}U) - \frac{1}{2}\frac{\partial g_{tt}}{\partial x^i}\mathrm{d}x^i. \tag{36}$$

Applying Eq.(30) here and noticing that $g_{tt}|_{U=0} = 0$, we obtain that

$$\nabla_{\mathbf{K}}\tilde{\mathbf{K}}\Big|_{U=0} = \kappa^2 GV\,\mathrm{d}U = \kappa\,\tilde{\mathbf{K}}\Big|_{U=0}, \qquad \nabla_{\mathbf{K}}\tilde{\mathbf{K}}\Big|_{V=0} = \kappa^2 GU\,\mathrm{d}V = -\kappa\,\tilde{\mathbf{K}}\Big|_{V=0}. \tag{37}$$

Thus it is confirmed that κ *is* the surface gravity.

Note that points where $U = V = 0$ are also admitted in the spacetime in this case. Such points are contained in a submanifold of $\mathcal{M}$, which is homeomorphic to the manifold $\mathcal{N}$. They are where the Killing vector field $\mathbf{K}$ vanishes and where the Killing horizons extend out, hence the submanifold is the bifurcate manifold, not necessarily a sphere because we didn't assume that for $\mathcal{N}$.

6 Discussion and Conclusions

For a stationary spacetime $(\mathcal{M}, \mathbf{g})$ that admitting the Killing vector field $\mathbf{K}$, the function $g_{tt} = \mathbf{g}(\mathbf{K}, \mathbf{K})$ is a globally defined intrinsic function which may have some zero points as the special points of the spacetime. For the spacetimes as we have discussed in this paper, the Regge-Wheeler tortoise function r_* is also intrinsic except that it is, generally speaking, not globally defined. The maximal domain in which it can be defined smoothly is a region that is often called the exterior region, the black hole region or the white hole region, and so on. If the tortoise function r_* can not be extended smoothly to cover the whole spacetime, there must be another tortoise function whose domain can't be enlarged to intersect with that of r_*. Hence the spacetime possesses regions, each with a Regge-Wheeler tortoise function defined smoothly in it and not able to be extended smoothly anymore. Since the spacetime we considered is connected, these regions must be separated by event horizons, on which the Killing vector field is lighlike, or, equivalently, the globally defined function g_{tt} vanishes on them. Now that spacetime is static, these event horizons are Killing horizons, with the surface gravity being constant and nonzero.

I. Rácz and R. M. Wald[13] have studied a globally hyperbolic stationary spacetime containing a black hole but not a white hole, with the event horizon of the black hole being a Killing horizon with compact cross sections. They proved that, if the surface gravity is nonzero and constant throughout the Killing horizon, the spacetime can be globally extended such that the (image of the) horizon is a proper subset of a regular bifurcate Killing horizon in the enlarged spacetime. The spacetime we considered in this paper is, of course, different from what is studied by I. Rácz and R. M. Wald, but we should be aware of the similarity between the results. Both the cases have implied that the constancy and nonzero property of the surface gravity are closely related to the extendibility of the spacetime. For the extremal Reissner-Nordström spacetime, for example, the surface gravity is zero everywhere on the horizon, the spacetime regions could not be extended as what had been done with the non-extremal case.

Of course, we have not fulfilled the tasks which were put forward in the beginning of this paper. However, the narrowered study has covered most of the well known examples of static

spacetimes, from which we find that the horizons, homeomorphic to the manifold $\mathcal{N}$, can be of arbitrary shape, including those with the volume not finite.

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References

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Co., San Fransisco, 1973.

[2] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge, 1973.

[3] R. M. Wald, *General Relativity*, The University of Chicago Press, Chicago, 1984.

[4] V. Frolov and I. Novikov, *Physics of Black Holes*, Kluwer, Dordrecht, 1998.

[5] T. Damour and R. Ruffini, *Phys. Rev.* **D14** (1976), 332.

[6] Z. Zhao and Y.-X. Gui, *IL Nuovo Cimento*, **109B** (1994), 355; Z. Zhao and X.-X. Dai, *Chin. Phys. Lett.*, **8** (1991), 548; Z. Zhao, Z. Luo and X.-X. Dai, *IL Nuovo Cimento*, **109B** (1994), 483; Z. Zhao and J.-Y. Zhu, *Inter. J. Theor. Phys.*, **33** (1994), 2147; Z.-h. Li and Z. Zhao, *IL Nuovo Cimento*, **110B** (1995) No. 12, 1427.

[7] R. K. Sachs and H. Wu, *General Relativity for Mathematicians*, Springer-Verlag, New York, 1977.

[8] J. Ehlers, *Survey of General Relativity Theory* in *Relativity, Astrophysics and Cosmology,* ed by W. Israel, R. Reidel, Dordrecht, 1973.

[9] P. K. Townsend, *Black Holes, lecture notes*, `gr-qc/9707012`.

[10] J. M. Bardeen, B. Carter and S. W. Hawking, *Commun. Math. Phys.*, **31** (1973), 161.

[11] R. Penrose and W. Rindler, *Spinors and Space-Time*, Vol. **1**: *Two-Spinor Calculus and Relativistic Fields*, Cambridge University Press, Cambridge, 1984.

[12] M. D. Kruskal, *Phys. Rev.*, **119** (1960), 1743.

[13] István Rácz and Robert M. Wald, *Global Extensions of Spacetimes Describing Asymptotic Final States of Black Holes*, `gr-qc/9507055`.